$\boxed{\text{MIT OCW GR PSET 8}}$

$\boxed{1}$. In lecture we derived the following formula for the leading gravitational radiation generated by a source:

$$h_{ij}^{TT} = \frac{2G}{r} \ddot{I}_{k\ell} \left(P_{ik} P_{j\ell} - \frac{1}{2} P_{ij} P_{k\ell} \right) \star$$

where we still use Einstein summation convention but ignore the distinction between upper + lower indices. Show that $\star$ still holds with $I_{k\ell}$ replaced with $\mathcal{I}_{k\ell}$ where $\mathcal{I}_{k\ell} \equiv I_{k\ell} - \frac{1}{3} \delta_{k\ell} I$ and $I = \delta_{k\ell} I_{k\ell}$:

$$\mathcal{I}_{k\ell} = I_{k\ell} - \frac{1}{3} \delta_{k\ell} \delta_{k\ell} I_{k\ell}$$
$$= I_{k\ell} \left(1 - \frac{1}{3} \delta_{k\ell} \delta_{k\ell} \right)$$

· So in theory our proof should show that:

$$-\frac{1}{3} \delta_{k\ell} \delta_{k\ell} \left(P_{ik} P_{j\ell} - \frac{1}{2} P_{ij} P_{k\ell} \right) = 0$$

P_{ij} is defined as $P_{ij} = \delta_{ij} - \frac{1}{2} \hat{n}_i \hat{n}_j$

$\longrightarrow$

where $\hat{n}_i$ is the unit vector pointing from the source of gravitational radiation to the observer.

$$\rightarrow \delta_{ij} P_{ij} = \delta_{ij}(\delta_{ij} - n_i n_j)$$

$$= \delta_{ij}\delta_{ij} - n_i n_i$$

$$= 3 - n_i n_i$$

$$= 2$$

← This term represents the sum of the squares of the components of the unit vector. By definition this should equal 1. E.g. if $\hat{n} = \hat{z}$ then $n_i n_i = 0^2 + 0^2 + 1^2 = 1$ ✓

- So back to the expression we want to prove is equal to zero:

$$\left(-\frac{1}{3}\delta_{kl}\delta_{kl}\right)\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)$$

$$= -\frac{1}{3}\delta_{kl}\underbrace{\delta_{kl}P_{ik}}_{P_{il}}P_{jl} + \frac{1}{6}P_{ij}\underbrace{P_{kl}\delta_{kl}}_{2}\delta_{kl}$$

$$= \delta_{kl}\left(-\frac{1}{3}P_{il}P_{jl} + \frac{1}{3}P_{ij}\right)$$

- So what does $P_{il}P_{jl}$ equal?

$$P_{il}P_{jl} = (\delta_{il} - n_i n_l)(\delta_{jl} - n_j n_l)$$

$$= \delta_{il}\delta_{jl} + n_i n_j \underbrace{n_l n_l}_{1} - \delta_{jl} n_i n_l - \delta_{il} n_j n_l$$

$$= \delta_{ij} + \cancel{n_i n_j} - \cancel{n_i n_j} - n_i n_j$$

$$= \delta_{ij} - n_i n_j$$

$$\equiv P_{ij}$$

- Intuitively this makes sense. P_{jl} picks out the jth component out of the lth component. P_{il} projects the ith component out of the lth component. Therefore, their composition, projects the ith component out of the jth component.
- Plugging this back in we get:

$$\left(-\tfrac{1}{3}\delta_{kl}\delta_{kl}\right)\left(P_{ik}P_{jl} - \tfrac{1}{2}P_{ij}P_{kl}\right) = \delta_{kl}\left(-\tfrac{1}{3}P_{ij} + \tfrac{1}{3}P_{ij}\right) = 0$$

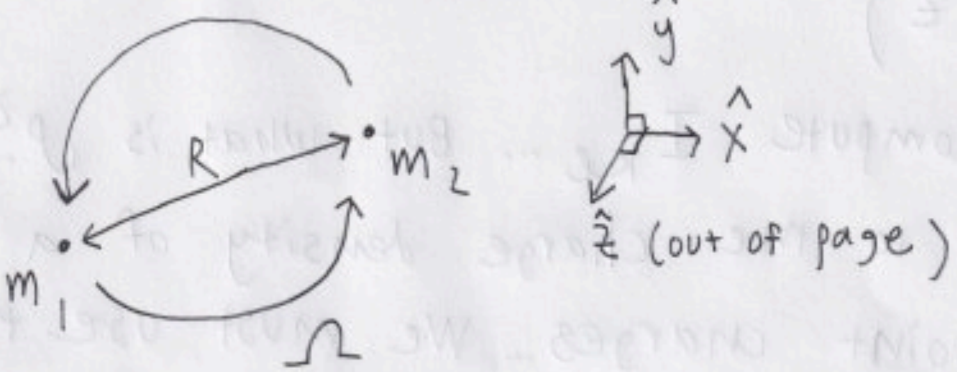
$$\rightarrow h_{ij}^{TT} = \frac{2G}{r}\ddot{I}_{kl}\left(P_{ik}P_{jl} - \tfrac{1}{2}P_{ij}P_{kl}\right)$$

$$= \frac{2G}{r}\ddot{I}_{kl}\left(P_{jk}P_{il} - \tfrac{1}{2}P_{ij}P_{kl}\right)$$

are equivalent

2 Binary System

- Consider a binary system consisting of 2 masses m_1 and m_2 in a circular orbit of radius R about one another. Consider the orbit to be adequately described using Newtonian gravity:



a · Compute h_{ij}^{TT} as measured by an observer looking down the $\hat{z}$ axis:

- We know that:

$$h_{ij}^{TT} = \left(\frac{2G}{r}\right)\left(\ddot{I}_{k\ell}\right)\left(P_{\ell i} P_{kj} - \frac{1}{2} P_{\ell k} P_{ij}\right)$$

where $I_{k\ell} \equiv \int_{\mathbb{R}^3} T_{00}\, x_k x_\ell \, dV \approx \int_{\mathbb{R}^3} \rho\, x_k x_\ell \, dV$

where ρ is the matter/energy density of the system...

$$-\text{and}- \quad P_{ij} \equiv \delta_{ij} - \hat{n}_i \hat{n}_j$$

where $\hat{n}$ is the unit vector from the gravitational source to the observer (in our case this is $\hat{z}$)

· Let's first compute I_{kl}... But what is ρ?
This is similar to the charge density of a collection of point charges... We must use the kronecker delta:

$$\rho(t) = m_1 \delta^3(\bar{r} - \bar{r}_1(t)) + m_2 \delta^3(\bar{r} - \bar{r}_2(t))$$

arbitrary position vector...

position of m_1 as a function of time

position of m_2 as a function of time

· Also enforce that the position of the COM lies at the origin:

$$\vec{R}_{COM} = 0$$

$$\Rightarrow \vec{R}_{com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = 0$$

$$\Rightarrow \vec{r}_1 = -\frac{m_2}{m_1}\vec{r}_2 \quad \text{and} \quad \vec{r}_2 = \frac{-m_1}{m_2}\vec{r}_1$$

- Define the difference in positions between the 2 masses as $\vec{r}$:

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

$$\Rightarrow \vec{r}_2 = \vec{r}_1 - \vec{r} \quad \text{and} \quad \vec{r}_1 = \vec{r}_2 + \vec{r}$$

$$\Rightarrow \vec{r}_1 = \frac{-m_2}{m_1}(\vec{r}_1 - \vec{r})$$

$$\Rightarrow \vec{r}_1(1 + m_2/m_1) = \frac{m_2}{m_1}\vec{r}$$

$$\Rightarrow \vec{r}_1 = \frac{m_2\vec{r}}{M} \quad \text{where } M \equiv m_1 + m_2$$

and $\vec{r}_2 = -m_1\vec{r}/M$

- Now we can actually compute the integral for I_{ke}

$$I_{k\ell} = \int_{\mathbb{R}^3} dV \left[m_1 \delta^3(\vec{r} - \vec{r}_1(t)) + m_2 \delta^3(\vec{r} - \vec{r}_2(t)) \right] x_k x_\ell$$

$$\Rightarrow I_{xx} = m_1 (\vec{r}_1(t) \cdot \hat{x})^2 + m_2 (\vec{r}_2(t) \cdot \hat{x})^2$$

- we just found previously $\vec{r}_1 = m_2 \vec{r} / M$ and $\vec{r}_2 = -m_1 \vec{r} / M$ but how do these vary with time? well assuming Newtonian theory to leading order $\vec{r}$ is a phasor with magnitude "R" and x, y, z components: $[\cos(\Omega t), \sin(\Omega t), 0]$ where $\Omega = \sqrt{GM/R^3}$ is the Newtonian precession:

i.e.

$$\vec{r}_1(t) = \frac{m_2}{M} \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{pmatrix} R \; ; \; \vec{r}_2(t) = \frac{-m_1}{M} \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{pmatrix} R$$

$$\rightarrow I_{xx} = \frac{m_1 m_2^2 R^2}{M^2}\cos^2(\Omega t) + \frac{m_2 m_1^2 R^2}{M^2}\cos^2(\Omega t)$$

$$\rightarrow I_{xx} = \frac{m_1 m_2 R^2}{M^2}\cos^2(\Omega t)\underbrace{(m_1 + m_2)}_{M}$$

$$\rightarrow I_{xx} = \mu R^2 \cos^2(\Omega t)$$

<u>Similarly</u>, $I_{yy} = \mu R^2 \sin^2(\Omega t)$

$$I_{xy} = I_{yx} = \mu R^2 \sin(\omega t)\cos(\omega t)$$

- all the components $I_{iz} = I_{zi} = 0$ since $\vec{r}_1 \cdot \hat{z} = \vec{r}_2 \cdot \hat{z} = 0$... So overall the matrix representation of I_{ij} is:

$$[I_{ij}] = \mu R^2 \begin{bmatrix} \cos^2(\Omega t) & \sin(\Omega t)\cos(\Omega t) & 0 \\ \sin(\Omega t)\cos(\Omega t) & \sin^2(\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now we need to take two time derivatives of each component, and we make the adiabatic assumption that $dR/dt = 0$ to Newtonian order:

$$\rightarrow \ddot{I}_{ij} = \nu R^2 \begin{bmatrix} -2\Omega^2 \cos(2\Omega t) & -2\Omega^2 \sin(2\Omega t) & 0 \\ -2\Omega^2 \text{[redacted]} \sin(2\Omega t) & 2\Omega^2 \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \ddot{I}_{ij} = -2\nu\Omega^2 R^2 \begin{bmatrix} \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now to find $h^{TT}_{ij} = \left(\frac{2G}{r}\right)\left(\ddot{I}_{k\ell}\right)\left(P_{ki}P_{\ell j} - \frac{1}{2}P_{ij}P_{k\ell}\right)$ we need to compute $P_{ij} = \delta_{ij} - \hat{n}_i\hat{n}_j = \delta_{ij} - \delta_{zz}$

$$\rightarrow [P_{ij}] = \text{diagonal}(1, 1, 0)$$

[redacted]

$$\rightarrow h^{TT}_{xx} = \left(\frac{2G}{r}\right)\left(\ddot{I}_{k\ell}\right)\left(P_{xk}P_{x\ell} - \frac{1}{2}P_{xx}P_{k\ell}\right)$$

$$= \left(\frac{2G}{r}\right)\Big(P_{xx}^2\ddot{I}_{xx} + \ddot{I}_{xy}P_{xx}P_{yy} + \ddot{I}_{yx}P_{xy}^2 - \frac{1}{2}\ddot{I}_{xx}P_{xx}^2$$

$$- \frac{1}{2}\ddot{I}_{yy}P_{xx}P_{yy} - \frac{1}{2}\cdot 2\ddot{I}_{xy}P_{xx}P_{xy}\Big)$$

$$= \left(\frac{2G}{r}\right)\left(\ddot{I}_{xx} + \ddot{I}_{xy} - \frac{1}{2}\ddot{I}_{xx} - \frac{1}{2}\ddot{I}_{yy}\right)$$

$$\rightarrow h^{TT}_{xx} = \left(\frac{G}{r}\right)\left(\ddot{I}_{xx} + 2\ddot{I}_{xy} - \ddot{I}_{yy}\right)$$

distance between orbiting masses

$$= \left(\frac{G}{r}\right)\left(2\nu\Omega^2 R^2\right)\left(-\cos(2\Omega t) - \cos(2\Omega t) - 2\sin(2\Omega t)\right)$$

distance of observer from COM

$$\rightarrow \boxed{h^{TT}_{xx} = \left(\frac{-4\nu G\Omega^2 R^2}{r}\right)\left(\sin(2\Omega t) + \cos(2\Omega t)\right)}$$

- We know h^{TT}_{zz} must equal zero since $P_{iz} = P_{zi} = 0$

$$\rightarrow h^{TT}_{yy} = -h^{TT}_{xx}$$

- Also all $h^{TT}_{zi} = h^{TT}_{iz} = 0$ for same reasoning
- So now we just need to find $h^{TT}_{xy} = h^{TT}_{yx}$:

$$h^{TT}_{xy} = \left(\frac{2G}{r}\right)\Big(\ddot{I}_{xx} P^2_{xy} - \frac{1}{2} P_{xx} P_{xy} \ddot{I}_{xx} + \ddot{I}_{yy} P_{yy} P_{xy}$$

$$- \frac{1}{2}\ddot{I}_{yy} P_{xy} P_{yy} + \ddot{I}_{xy} P_{xx} P_{yy} - \frac{1}{2} P_{xy} P_{xy} + \cdots$$

$$+ \ddot{I}_{yx} P_{xy}^{2} - \frac{1}{2} P_{xy} P_{xy} \ddot{I}_{yx} \Big)$$

$$\rightarrow h^{TT}_{xy} = \left(\frac{2G}{r}\right)\ddot{I}_{xy} = \frac{-4\nu G \Omega R^{2}}{r} \sin(2\Omega t)$$

. So overall we get that:

$$\boxed{\left[h^{TT}_{ij}\right] = \left(\frac{-4\nu G \Omega^{2} R^{2}}{r}\right) \begin{bmatrix} \sin(2\Omega t) + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & -\sin(2\Omega t) - \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\boxed{6}$. compute the rate at which energy is carried away from the system by Gravitational waves:-

. The formula to do this wasn't actually given in the Prof. Hughes' presentation up to this point in the lecture series, but we will use the following equation:

← from Wikipedia...

$$\frac{dE}{dt}_{GW} \equiv \dot{E}_{GW} = \frac{G}{5}\left\langle \dddot{I}_{ij} \dddot{I}_{ij} \right\rangle$$ where those are triple dots + $\langle \cdot \rangle$ represents a time average over one cycle ⟶

$$\rightarrow \dot{E}_{GW} = \left(\frac{G}{5}\right)\left(\langle \dddot{I}_{xx} \dddot{I}_{xx} \rangle + \langle \dddot{I}_{yy} \dddot{I}_{yy} \rangle + 2\langle \dddot{I}_{xy} \dddot{I}_{xy} \rangle \right)$$

<u>Aside</u>

$$\dddot{I}_{xx} = 4\nu\Omega^3 R^2 \sin(2\Omega t)$$

$$\dddot{I}_{yy} = -4\nu\Omega^3 R^2 \sin(2\Omega t)$$

$$\dddot{I}_{xy} = -4\nu\Omega^3 R^2 \cos(2\Omega t)$$

- Recall the time average of sin / cos over one period is 1/2, so we get that:

$$\dot{E}_{GW} = \left(\frac{G}{5}\right)\left(16\nu^2\Omega^6 R^4\right)\left(\frac{1}{2} + \frac{1}{2} + \frac{2}{2}\right)$$

and $\Omega = \sqrt{GM/R^3}$

$$\rightarrow \dot{E}_{GW} = 32\nu^2 G^4 M^3 R^4 / 5R^9$$

$$\rightarrow \boxed{\dot{E}_{GW} = \frac{32\nu^2 M^3 G^4}{5R^5}}$$

is the rate of energy lost + radiated by the gravitational waves due to this loss of energy, the radius of the orbit will gradually shrink and the frequency of the binary will "chirp" to higher frequencies as time passes...

$\boxed{c}$ · Use global conservation of energy to find $\frac{dr}{dt}$

· Assert that $\frac{d}{dt}\left(\underbrace{E_{kinetic}}_{\equiv T} + \underbrace{E_{potential}}_{\equiv V} + E_{GW}\right) = 0$

· By the virial theorem for circular Newtonian orbits the time average of T + V are related via : $\langle T \rangle = -\frac{1}{2}\langle V \rangle$

$$\rightarrow \frac{d}{dt}\left(\frac{\langle V \rangle}{2}\right) + \dot{E}_{GW} = 0$$

$V(r)$ is given by $\dfrac{-Gm_1m_2}{r}$

$$\rightarrow \langle \dot{V} \rangle = \frac{Gm_1m_2}{r^2} \cdot \frac{dr}{dt}$$

$$\rightarrow \frac{dr}{dt} = \frac{-2r^2}{Gm_1m_2}\dot{E}_{GW}$$

$$\rightarrow \frac{dr}{dt} = \frac{-2r^2}{G\underbrace{m_1 m_2}_{\nu M}} \cdot \frac{32\nu^2 M^3 G^4}{5r^5}$$

$$\rightarrow \boxed{\frac{dr}{dt} = \frac{-64\nu M^2 G^3}{5r^3}}$$

$\boxed{d}$ · Now derive $d\Omega/dt$:

· We can simply use the chain rule here:

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial r} \cdot \frac{dr}{dt} \quad ; \quad \Omega = \sqrt{GM}\, r^{-3/2}$$

$$\rightarrow \partial_r \Omega = \frac{-3\sqrt{GM}}{2} r^{-5/2}$$

$$\rightarrow \boxed{\dot{\Omega} = \frac{96\nu M^{2.5} G^{3.5}}{5r^{5.5}}}$$

3 Wave Equation for Riemann Tensor in Linear GR

- Recall that the EFE can be written in the trace-reversed form:

$$R^{\nu}{}_{\alpha\nu\beta} = R_{\alpha\beta} = 8\pi \overline{T}_{\alpha\beta}$$

where $\overline{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\gamma}{}_{\gamma}$

- In this problem it will be important to keep track of relative horizontal ordering in both the upstairs + downstairs (this is an important part of all of tensor calculus in fact) so e.g. something like

$$R^{\nu}{}_{\alpha\nu\beta} \neq R_{\alpha}{}^{\nu}{}_{\nu\beta}$$

- Begin with the Bianchi identity to linear order in h as:

$$\partial_{\alpha} R_{\beta\gamma\nu\sigma} + \partial_{\beta} R_{\gamma\alpha\nu\sigma} + \partial_{\gamma} R_{\alpha\beta\nu\sigma} = 0 \quad \star$$

- In linear GR, not only can we raise the indices of $h_{\nu\sigma}$ with $\eta_{\nu\sigma}$, but this is true of other tensors as well such as the Riemann tensor. Taking advantage of this fact + that $\eta^{\nu\sigma}$ commutes with ∂_{β} we $\rightsquigarrow$

multiply ⊛ by $\eta^{\alpha\nu}$ to get:

$$\eta^{\alpha\nu}\partial_\alpha R_{\beta\gamma\nu\sigma} + \partial_\beta \eta^{\alpha\nu} R_{\gamma\alpha\nu\sigma} + \partial_\gamma \eta^{\alpha\nu} R_{\alpha\beta\nu\sigma} = 0$$

· Use $R_{\beta\gamma\nu\sigma} = R_{\nu\sigma\beta\gamma}$

on first term

and $R_{\gamma\alpha\nu\sigma} = -R_{\alpha\gamma\nu\sigma}$ on 2nd term

$$\rightarrow \partial^\nu R_{\nu\sigma\beta\gamma} - \partial_\beta \underbrace{R^\nu{}_{\gamma\nu\sigma}}_{\substack{\text{III} \\ 8\pi \overline{T}_{\gamma\sigma}}} + \partial_\gamma \underbrace{R^\nu{}_{\beta\nu\sigma}}_{\substack{\text{III} \\ 8\pi \overline{T}_{\beta\sigma}}} = 0$$

flip upper + lower since dummies

with $G = 1$

$$\rightarrow \boxed{\partial_\nu R^\nu{}_{\sigma\beta\gamma} = \left(\partial_\beta \overline{T}_{\gamma\sigma} - \partial_\gamma \overline{T}_{\beta\sigma}\right) 8\pi G} \quad (*)$$

$\boxed{6}$ · Now again use the Bianchi identity and (*) to develop a wave equation of the form:

$$\Box R_{\alpha\beta\nu\sigma} = 8\pi G \left[\text{term with double gradients of } \overline{T}_{\nu\sigma}\right]$$

• Start with the Bianchi identity again:

$$\partial_\alpha R_{\beta\gamma\nu\sigma} + \partial_\beta R_{\gamma\alpha\nu\sigma} + \partial_\gamma R_{\alpha\beta\nu\sigma} = 0$$

• apply ∂^α to all terms:

$$\rightarrow \underbrace{\partial_\alpha \partial^\alpha}_{\substack{\text{III} \\ \Box \text{ d'Alemlertian}}} R_{\beta\gamma\nu\sigma} + \partial_\beta \partial^\alpha \underbrace{R_{\gamma\alpha\nu\sigma}}_{\substack{\shortparallel \\ -R_{\alpha\gamma\nu\sigma}}} + \partial_\gamma \underbrace{\partial^\alpha R_{\alpha\beta\nu\sigma}}_{\text{previously derived formula in part (a)...}} = 0$$

$$\rightarrow \Box R_{\beta\gamma\nu\sigma} = +\partial_\beta \partial^\alpha R_{\alpha\gamma\nu\sigma} - \partial_\gamma \partial^\alpha R_{\alpha\beta\nu\sigma}$$

$$= \partial_\beta \left(\partial_\nu \overline{T}_{\sigma\gamma} - \partial_\sigma \overline{T}_{\nu\gamma} \right)$$

$$- \partial_\gamma \left(\partial_\nu \overline{T}_{\sigma\beta} - \partial_\sigma \overline{T}_{\nu\beta} \right)$$

$$\rightarrow \boxed{\begin{aligned} \Box R_{\beta\gamma\nu\sigma} = \partial_\nu \partial_\beta \overline{T}_{\sigma\gamma} + \partial_\gamma \partial_\sigma \overline{T}_{\nu\beta} \\ - \partial_\nu \partial_\gamma \overline{T}_{\sigma\beta} - \partial_\sigma \partial_\beta \overline{T}_{\nu\gamma} \end{aligned}} \quad (\star)$$

• Is the wave equation for the Riemann Tensor in Linear GR ✓✓ :)

• Now we need to solve this wave equation using a radiative Green's function:

• The "radiative Green's function" is:

$$G(t, \underset{\sim}{x}; t', \underset{\sim}{x}') = \frac{-\delta[t' - (t - |\underset{\sim}{x} - \underset{\sim}{x}'|)]}{4\pi |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

• We have the diff eq ⊛ implying that the Riemann tensor is given by the following integral:

$$\boxed{R_{\beta\gamma\nu\sigma} = \int dt' \int d^3x' \left[-\partial_\nu \partial_\beta \bar{T}_{\sigma\gamma} - \partial_\gamma \partial_\sigma \bar{T}_{\nu\beta} + \partial_\nu \partial_\gamma \bar{T}_{\sigma\beta} + \partial_\sigma \partial_\beta \bar{T}_{\nu\gamma} \right] \cdot \frac{\delta[t' - (t - |\underset{\sim}{x} - \underset{\sim}{x}'|)]}{4\pi |\underset{\sim}{x} - \underset{\sim}{x}'|}}$$

• Since [redacted] we aren't actually given an explicit form for $\bar{T}_{\alpha\beta}$, I think this is the most we can do (no explicit calculations...) W

c • Now specialize to a plane gravitational wave propagating in the z-direction through vacuum. The corresponding solution to the previous wave equation is $R_{\alpha\beta\nu\sigma} = R_{\alpha\beta\nu\sigma}(t - z)$.

· Use the Bianchi identity to show that the only non-zero Rieman components are R_{i0j0} + the related symmetries:

<u>1st</u> What components are possible?

$$R_{\alpha\beta\mu\nu} \rightarrow \begin{cases} R_{00\mu\nu} \; (\star) \\ R_{i0\mu\nu} = -R_{0i\mu\nu} \\ R_{ij\mu\nu} \; (\star\star) \end{cases}$$

where i,j are spatial indices, but μ,ν still range from t,x,y,z ...

· First find a relation for components like $(\star)$

· since $R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}(t-z)$

$$\rightarrow \partial_t R_{\alpha\beta\mu\nu} = -\partial_z R_{\alpha\beta\mu\nu}$$

· Bianchi identity states:

$$\partial_\alpha R_{\beta\gamma\mu\nu} + \partial_\beta R_{\gamma\alpha\mu\nu} + \partial_\gamma R_{\alpha\beta\mu\nu} = 0$$

· choose $\alpha = \beta = 0 = t$ and $\gamma = z$

$$\rightarrow \partial_t R_{tz\mu\nu} + \partial_t R_{zt\mu\nu} + \partial_z R_{tt\mu\nu} = 0$$

- Use Riemann symmetry to set $R_{tz\nu\sigma} = -R_{zt\nu\sigma}$ and use $\partial_z \rightarrow -\partial_t$. Then we get that:

$$\partial_t \left(R_{tz\nu\sigma} - R_{tz\nu\sigma} - R_{tt\nu\sigma} \right) = 0$$

$$\rightarrow \partial_t (0) = \partial_t \left(R_{tt\nu\sigma} \right)$$

$$\rightarrow R_{tt\nu\sigma} = 0$$

- In general we could also say it equals a constant but since we are dealing with gravitational waves which must decay to zero @ ∞, that constant $= 0$ ...
- Now what about $R_{ij\nu\sigma}$? equation ★★ ...
- Start with Bianchi:

$$\partial_\alpha R_{\beta\gamma\nu\sigma} + \partial_\beta R_{\gamma\alpha\nu\sigma} + \partial_\gamma R_{\alpha\beta\nu\sigma} = 0$$

choose $\alpha = i$, $\beta = j$, $\gamma = t$:

$$\rightarrow \partial_i R_{j0\nu\sigma} + \partial_j R_{ti\nu\sigma} + \partial_t R_{ij\nu\sigma} = 0$$

$$\rightarrow \left(\partial_i R_{j0\nu\sigma} - \partial_j R_{i0\nu\sigma} \right) + \partial_t R_{ij\nu\sigma} = 0$$

- Now multiply through by η^{ii}

$$\rightarrow \underbrace{\eta^{ij}}_{\substack{\text{sym} \\ \text{under} \\ i \to j}} \big(\underbrace{\partial_i R_{j0\mu\nu} - \partial_j R_{i0\mu\nu}}_{\substack{\text{anti-sym} \\ \text{under} \\ i \to j}}\big) + \eta^{ij} \partial_t R_{ij\mu\nu} = 0$$

$$\rightarrow \eta^{ij} \partial_t R_{ij\mu\nu} = 0$$

- Multiply through by inverse $[\eta^{ij}]^{-1}$

$$\rightarrow \partial_t R_{ij\mu\nu} = 0 \longrightarrow R_{ij\mu\nu} = 0$$

- Going back to our diagram of Riemann components we now have:

$R_{\alpha\beta\mu\nu}$
- $\rightarrow R_{00\mu\nu} = 0$ X
- $\rightarrow R_{ij\mu\nu} = 0$ X
- $\rightarrow R_{i0\mu\nu}$
  - $\rightarrow R_{i000} = 0$ via sym.
  - $\rightarrow R_{i0jk} = 0$ via sym.
  - $\rightarrow R_{i0j0} = ?$ Hmm...

$$\boxed{\text{· So only } R_{i0j0} \text{ and their related symmetries are possibly non-zero } \checkmark}$$

$\boxed{d}$ · Show that the only non-zero R_{i0j0} are $R_{x0x0}(t-z) = -R_{y0y0}(t-z)$ and $R_{x0y0}(t-z) = R_{y0x0}(t-z)$. The first non-zero components correspond to the + polarization discussed in lecture; the 2nd corresponds to the x polarization...

· Start with Bianchi with first permutation term set to $\partial_0 R_{z0\mu\nu}$:

$\rightarrow \partial_0 R_{z0\mu\nu} + \partial_z R_{00\mu\nu} + \partial_0 R_{0z\mu\nu} = 0$

$\rightarrow \partial_0 (R_{z0\mu\nu} - R_{z0\mu\nu}) + \partial_z R_{00\mu\nu} = 0$

$\rightarrow \partial_z R_{00\mu\nu} = 0$

$\rightarrow \partial_z \delta^z_0 R_{z0\mu\nu} = 0$

$\rightarrow \partial_t R_{z0\mu\nu} = 0$

$\rightarrow R_{z0\mu\nu} = \text{constant} \rightarrow \emptyset$

· So all the $R_{z0\mu\nu} = R_{\mu\nu z0}$ and symmetries go to zero leaving only R_{x0x0}, R_{y0y0}, R_{x0y0}, + R_{y0x0}. It is obvious that $\boxed{R_{x0y0} = R_{y0x0}}$ via Riemann symmetry. To argue that $R_{x0x0} = -R_{y0y0}$ we must remember the problem states we are in vacuum with NO sources. This implies the Ricci Tensor $R_{\nu\gamma} = 0$ + the Ricci Tensor is the only non-zero trace of Riemann. Since the only non-zero diagonal components left are R_{x0x0} and R_{y0y0} + their sum must equal zero → $\boxed{R_{x0x0} = -R_{y0y0}}$

e. Define fields $h_+(t-z)$ and $h_\times(t-z)$ in terms of the Riemann components:

$$R_{x0x0} = -\frac{1}{2}\partial_t^2 h_+ \quad ; \quad R_{x0y0} = -\frac{1}{2}\partial_t^2 h_\times$$

and Remember in Linear GR:

$$R_{\alpha\beta\nu\gamma} = \frac{1}{2}\left(\partial_\alpha\partial_\gamma h_{\beta\nu} + \partial_\beta\partial_\nu h_{\alpha\gamma} - \partial_\alpha\partial_\nu h_{\beta\gamma} - \partial_\beta\partial_\gamma h_{\alpha\nu}\right)$$

- show that $\begin{cases} h_+ = h_{xx}^{TT} = -h_{yy}^{TT} \text{ and} \\ h_\times = h_{xy}^{TT} = h_{yx}^{TT} \end{cases}$

· Using the definition of $R_{\alpha\beta\mu\nu}$ we get that:

$$R_{x0x0} = -\frac{1}{2}\partial_0^2 h_+$$
$$= \frac{1}{2}\left(\partial_x\partial_0 h_{0x} + \partial_0\partial_x h_{x0} - \partial_x^2 h_{00} - \partial_0^2 h_{xx}\right)$$

· Since we define $h_+(t-z)$ as a function only of t, z we get that $\partial_x h_{ij} = \partial_y h_{ij} = 0$

$$\rightarrow R_{x0x0} = -\frac{1}{2}\partial_0^2 h_+ = -\frac{1}{2}\partial_0^2 h_{xx}$$

$$\rightarrow \boxed{h_+ = h_{xx}}$$

· Now use $R_{x0x0} = -R_{y0y0}$:

$$= \frac{1}{2}\partial_0^2 h_{yy} = -\frac{1}{2}\partial_0^2 h_+$$

$$\rightarrow \boxed{h_+ = -h_{yy} = h_{xx}}$$

· Now use the definition:

$$R_{x0y0} = -\frac{1}{2}\partial_t^2 h_x = \frac{1}{2}\left(\partial_x\partial_0 h_{0y} + \partial_0\partial_y h_{x0} - \partial_x\partial_y h_{00} - \partial_0^2 h_{xy}\right)$$

$\rightsquigarrow -\frac{1}{2}\partial_0^2 h_\times = -\frac{1}{2}\partial_0^2 h_{xy}$

$\rightsquigarrow h_{xy} = h_\times$ and since $R_{x0y0} = R_{y0x0}$

$$\Longrightarrow \boxed{h_\times = h_{xy} = h_{yx}}$$

[f]. Show that when one rotates the coordinate system about the waves' propagation direction (the z-axis in our case) by an angle θ (so that $x' + iy' = (x+yi)e^{-i\theta}$) then the gravitational-wave fields h_+ and $h_\times$ transform s.t:

$$h_+' + ih_\times' = (h_+ + ih_\times)e^{-i2\theta}$$

· This statement means that the graviton is spin-2:

· Recall how Tensors transform under transformations:

$$T_{\alpha'\beta'} = \frac{\partial x^\nu}{\partial x^{\alpha'}} \cdot \frac{\partial x^\gamma}{\partial x^{\beta'}} T_{\nu\gamma}$$

$$\longrightarrow h_{x'x'} = \frac{\partial x'}{\partial x} \cdot \frac{\partial x'}{\partial x} h_{xx} \quad ; \quad \frac{\partial x'}{\partial x} = e^{-i\theta}$$

$$\longrightarrow h_{x'x'} = e^{-i2\theta} h_{xx}$$

· Also

$$h_{x'y'} = \frac{\partial x'}{\partial x} \cdot \frac{\partial y'}{\partial y} h_{xy} \quad ; \quad \frac{\partial y'}{\partial y} = e^{-i\theta} \text{ also}$$

$$\longrightarrow h_{x'y'} = e^{-i2\theta} h_{xy}$$

$$\boxed{h_+' + i h_x' = h_{x'x'} + i h_{x'y'}}$$

$$\boxed{= e^{-i2\theta}(h_{xx} + i h_{xy})}$$

$$\boxed{= e^{-i2\theta}(h_+ + i h_x)}$$

Q.E.D.

4 . Nonlinear Wave equation for the Riemann Tensor

· Use the full Bianchi identity:

$$\nabla_\alpha R_{\beta\gamma\nu\sigma} + \nabla_\beta R_{\gamma\alpha\nu\sigma} + \nabla_\gamma R_{\alpha\beta\nu\sigma} = 0$$

a . Develop the fully covariant analog to part 3.a:

$$\nabla_\alpha R^{\wedge}{}_{\beta\gamma\delta} = 8\pi G(\text{covariant gradients of } T_{\mu\nu})$$

· apply $g^{\alpha\nu}$ to both sides + remember that we have metric compatibility s.t. $\nabla_\gamma g_{\alpha\beta} = 0$

$$\rightarrow \nabla^{\mu} R_{\mu\nu\beta\gamma} + \nabla_{\beta} g^{\alpha\mu} R_{\gamma\alpha\mu\nu} + \nabla_{\gamma} g^{\alpha\mu} R_{\alpha\beta\mu\nu} = 0$$

$$\rightarrow \nabla_{\mu} R^{\mu}{}_{\nu\beta\gamma} - \nabla_{\beta} R^{\mu}{}_{\gamma\mu\nu} + \nabla_{\gamma} R^{\mu}{}_{\beta\mu\nu} = 0$$

$$\rightarrow \nabla_{\mu} R^{\mu}{}_{\nu\beta\gamma} = \nabla_{\beta} R^{\mu}{}_{\gamma\mu\nu} - \nabla_{\gamma} R^{\mu}{}_{\beta\mu\nu}$$

$$= \nabla_{\beta} R_{\gamma\nu} - \nabla_{\gamma} R_{\beta\nu}$$

$$\boxed{\nabla_{\mu} R^{\mu}{}_{\nu\beta\gamma} = 8\pi G \left(\nabla_{\beta} \overline{T}_{\gamma\nu} - \nabla_{\gamma} \overline{T}_{\beta\nu} \right)}$$

. which just looks like what we previously found in linear GR but $\partial_{\alpha} \rightarrow \nabla_{\alpha}$. However since $[\nabla_{\alpha}, \nabla_{\beta}] \neq 0$, when we try to derive the corresponding wave-equation next, it will be more juicy ...

[b]. Now find an expression for $\Box R_{\alpha\beta\gamma\delta}$ ...

This is known as the Penrose equation for Roger Penrose ...

$\rightsquigarrow$

• Start with covariant Bianchi identity:

$$\nabla_\alpha R_{\beta\gamma\nu\sigma} + \nabla_\beta R_{\gamma\alpha\nu\sigma} + \nabla_\gamma R_{\alpha\beta\nu\sigma} = 0$$

• Apply ∇^α to left of each term

$$\nabla^\alpha\nabla_\alpha R_{\beta\gamma\nu\sigma} + \nabla^\alpha\nabla_\beta R_{\gamma\alpha\nu\sigma} + \nabla^\alpha\nabla_\gamma R_{\alpha\beta\nu\sigma} = 0$$

$$\rightarrow \Box R_{\beta\gamma\nu\sigma} - \nabla_\alpha\nabla_\beta R^\alpha{}_{\gamma\nu\sigma} + \nabla_\alpha\nabla_\gamma R^\alpha{}_{\beta\nu\sigma} = 0$$

$$\rightarrow \Box R_{\beta\gamma\nu\sigma} = \nabla_\alpha\nabla_\beta R^\alpha{}_{\gamma\nu\sigma} - \nabla_\alpha\nabla_\gamma R^\alpha{}_{\beta\nu\sigma}$$

$$= [\nabla_\alpha, \nabla_\beta] R^\alpha{}_{\gamma\nu\sigma} - [\nabla_\alpha, \nabla_\gamma] R^\alpha{}_{\beta\nu\sigma}$$

$$+ \nabla_\beta \underbrace{\nabla_\alpha R^\alpha{}_{\gamma\nu\sigma}}_{\text{previously derived formula in 4.a.}} - \nabla_\gamma\nabla_\alpha R^\alpha{}_{\beta\nu\sigma}$$

$$= [\nabla_\alpha, \nabla_\beta] R^\alpha{}_{\gamma\nu\sigma} - [\nabla_\alpha, \nabla_\gamma] R^\alpha{}_{\beta\nu\sigma}$$

$$+ \nabla_\beta(\nabla_\nu \overline{T}_{\sigma\gamma} - \nabla_\sigma \overline{T}_{\nu\gamma})\, 8\pi G$$

$$- 8\pi G \nabla_\gamma(\nabla_\nu \overline{T}_{\sigma\beta} - \nabla_\sigma \overline{T}_{\nu\beta})$$

· In general, $[\nabla_\nu, \nabla_\sigma] T^{\alpha\dots}_{\beta\dots}$

$$= \sum_{\substack{\text{upper}\\\text{indices}}} R^{\alpha}_{\ \lambda\nu\sigma} T^{\lambda\dots}_{\beta\dots} + \sum_{\substack{\text{lower}\\\text{indices}}} R^{\lambda}_{\ \beta\nu\sigma} T^{\alpha\dots}_{\lambda\dots}$$

· So the Riemann terms with $[\nabla_\alpha, \nabla_\beta]$ commutators in front of them can be expanded like:

$$\begin{aligned}\Box R_{\beta\gamma\nu\sigma} = {} & R^{\alpha}_{\ \lambda\alpha\beta} R^{\lambda}_{\ \gamma\nu\sigma} - R^{\lambda}_{\ \gamma\alpha\beta} R^{\alpha}_{\ \lambda\nu\sigma} \\ & - R^{\lambda}_{\ \nu\alpha\beta} R^{\alpha}_{\ \gamma\lambda\sigma} - R^{\lambda}_{\ \sigma\alpha\beta} R^{\alpha}_{\ \gamma\nu\lambda} \\ & - R^{\alpha}_{\ \lambda\alpha\gamma} R^{\lambda}_{\ \beta\nu\sigma} + R^{\lambda}_{\ \beta\alpha\gamma} R^{\alpha}_{\ \lambda\nu\sigma} \\ & + R^{\lambda}_{\ \nu\alpha\gamma} R^{\alpha}_{\ \beta\lambda\sigma} + R^{\lambda}_{\ \sigma\alpha\gamma} R^{\alpha}_{\ \beta\nu\lambda} \\ & + 8\pi G \left(\nabla_\beta \nabla_\nu \overline{T}_{\sigma\gamma} + \nabla_\gamma \nabla_\sigma \overline{T}_{\nu\beta} \right. \\ & \left. - \nabla_\beta \nabla_\sigma \overline{T}_{\nu\gamma} - \nabla_\gamma \nabla_\nu \overline{T}_{\sigma\beta} \right)\end{aligned}$$

· The two circled terms have sums over the 1st + third indices so they are ...

... equivalent to the Ricci tensor + therefore proportional to the trace-reversed stress energy tensor... Simplifying we get that:

$$\boxed{\begin{aligned}\Box R_{\beta\gamma\nu\sigma} = {} & 8\pi G\big(\bar{T}_{\lambda\beta} R^{\lambda}{}_{\gamma\nu\sigma} - \bar{T}_{\lambda\gamma} R^{\lambda}{}_{\beta\nu\sigma} \\ & + \nabla_\beta \nabla_\nu \bar{T}_{\sigma\gamma} + \nabla_\gamma \nabla_\sigma \bar{T}_{\nu\beta} - \nabla_\beta \nabla_\sigma \bar{T}_{\nu\gamma} \\ & - \nabla_\gamma \nabla_\nu \bar{T}_{\sigma\beta}\big) - R^{\lambda}{}_{\gamma\alpha\beta} R^{\alpha}{}_{\lambda\nu\sigma} \\ & - R^{\lambda}{}_{\nu\alpha\beta} R^{\alpha}{}_{\gamma\lambda\sigma} - R^{\lambda}{}_{\sigma\alpha\beta} R^{\alpha}{}_{\gamma\nu\lambda} \\ & + R^{\lambda}{}_{\beta\alpha\gamma} R^{\alpha}{}_{\lambda\nu\sigma} + R^{\lambda}{}_{\nu\alpha\gamma} R^{\alpha}{}_{\beta\lambda\sigma} \\ & + R^{\lambda}{}_{\sigma\alpha\gamma} R^{\alpha}{}_{\beta\nu\lambda}\end{aligned}}$$

- Which involves terms with two gradients of $\bar{T}_{\alpha\beta}$, terms like $\bar{T}$ multiplied by Riemann, + terms like Riemann times Riemann as our hint suggested Q.E.D W